

THE ANNIHILATING-SUBMODULE GRAPH OF MODULES OVER COMMUTATIVE RINGS II

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ABSTRACT. Let M be a module over a commutative ring R . In this paper, we continue our study of annihilating-submodule graph $AG(M)$ which was introduced in (The Zariski topology-graph of modules over commutative rings, Comm. Algebra., 42 (2014), 3283–3296). $AG(M)$ is a (undirected) graph in which a nonzero submodule N of M is a vertex if and only if there exists a nonzero proper submodule K of M such that $NK = (0)$, where NK , the product of N and K , is defined by $(N : M)(K : M)M$ and two distinct vertices N and K are adjacent if and only if $NK = (0)$. We prove that if $AG(M)$ is a tree, then either $AG(M)$ is a star graph or a path of order 4 and in the latter case $M \cong F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule. Moreover, we prove that if M is a cyclic module with at least three minimal prime submodules, then $gr(AG(M)) = 3$ and for every cyclic module M , $cl(AG(M)) \geq |Min(M)|$.

1. INTRODUCTION

Throughout this paper R is a commutative ring with a non-zero identity and M is a unital R -module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M .

Define $(N :_R M)$ or simply $(N : M) = \{r \in R \mid rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $Ann_R(M)$ or simply $Ann(M)$. M is said to be faithful if $Ann(M) = (0)$.

Let $N, K \leq M$. Then the product of N and K , denoted by NK , is defined by $(N : M)(K : M)M$ (see [6]).

There are many papers on assigning graphs to rings or modules (see, for example, [4, 7, 10, 11]). The annihilating-ideal graph $AG(R)$ was introduced and studied in [11]. $AG(R)$ is a graph whose vertices are ideals of R with nonzero annihilators and in which two vertices I and J are adjacent if and only if $IJ = (0)$. Later, it was modified and further studied by many authors (see, e.g., [1-3]).

In [7, 8], we generalized the above idea to submodules of M and defined the (undirected) graph $AG(M)$, called *the annihilating-submodule graph*, with vertices $V(AG(M)) = \{N \leq M \mid \text{there exists } (0) \neq K < M \text{ with } NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M \text{ with } (N : M) \neq Ann(M) \mid \text{there exists a submodule } K < M \text{ with } (K : M) \neq Ann(M) \text{ and } NK = (0)\}$. Note that M is a vertex of $AG(M)$ if and only if there exists a

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nonzero proper submodule N of M with $(N : M) = \text{Ann}(M)$ if and only if every nonzero submodule of M is a vertex of $AG(M)$.

In this work, we continue our studying in [7, 8] and we generalize some results related to annihilating-ideal graph obtained in [1-3] for annihilating-submodule graph.

A prime submodule of M is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [13].

The prime radical $\text{rad}_M(N)$ or simply $\text{rad}(N)$ is defined to be the intersection of all prime submodules of M containing N , and in case N is not contained in any prime submodule, $\text{rad}_M(N)$ is defined to be M [13].

The notations $Z(R)$, $\text{Nil}(R)$, and $\text{Min}(M)$ will denote the set of all zero-divisors, the set of all nilpotent elements of R , and the set of all minimal prime submodules of M , respectively. Also, $Z_R(M)$ or simply $Z(M)$, the set of zero divisors on M , is the set $\{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in G , denoted by $cl(G)$, is called the clique number of G . Let $\chi(G)$ denote the chromatic number of the graph G , that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$.

In section 2 of this paper, we prove that if $AG(M)$ is a tree, then either $AG(M)$ is a star graph or is the path P_4 and in this case $M \cong F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule (see Theorem 2.7). Next, we study the bipartite annihilating-submodule graphs of modules over Artinian rings (see Theorem 2.8). Moreover, we give some relations between the existence of cycles in the annihilating-submodule graph of a cyclic module and the number of its minimal prime submodules (see Theorem 2.18 and Corollary 2.19)).

Let us introduce some graphical notions and denotations that are used in what follows: A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \dots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i .

A graph H is a subgraph of G , if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V ; that is, U and V are each independent sets and complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m , respectively, and $E(G)$ connects every vertex in V with all vertices in U . Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of U . For every vertex $v \in V(G)$, the size of $N(v)$ is denoted by $d(v)$. If all the vertices of G have the same degree k , then G is called k -regular, or simply regular. An independent set is a subset of the vertices of a graph such that no vertices are adjacent. We denote by P_n and C_n , a path and a cycle of order n , respectively. Let G and G' be two graphs. A graph homomorphism from G to G' is a mapping $\phi : V(G) \rightarrow V(G')$ such that for every edge $\{u, v\}$ of G , $\{\phi(u), \phi(v)\}$ is an edge of G' . A retract of G is a subgraph H of G such that there exists a homomorphism $\phi : G \rightarrow H$ such

that $\phi(x) = x$, for every vertex x of H . The homomorphism ϕ is called the retract (graph) homomorphism (see [19]).

2. THE ANNIHILATING-SUBMODULE GRAPH II

An ideal $I \leq R$ is said to be nil if I consist of nilpotent elements; I is said to be nilpotent if $I^n = (0)$ for some natural number n .

Proposition 2.1. Suppose that e is an idempotent element of R . We have the following statements.

- (a) $R = R_1 \oplus R_2$, where $R_1 = eR$ and $R_2 = (1 - e)R$.
- (b) $M = M_1 \oplus M_2$, where $M_1 = eM$ and $M_2 = (1 - e)M$.
- (c) For every submodule N of M , $N = N_1 \times N_2$ such that N_1 is an R_1 -submodule M_1 , N_2 is an R_2 -submodule M_2 , and $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$.
- (d) For submodules N and K of M , $NK = N_1K_1 \times N_2K_2$ such that $N = N_1 \times N_2$ and $K = K_1 \times K_2$.
- (e) Prime submodules of M are $P \times M_2$ and $M_1 \times Q$, where P and Q are prime submodules of M_1 and M_2 , respectively.

Proof. This is clear. □

We need the following lemmas.

Lemma 2.2. (See [5, Proposition 7.6].) Let R_1, R_2, \dots, R_n be non-zero ideals of R . Then the following statements are equivalent:

- (a) ${}_R R = R_1 \oplus \dots \oplus R_n$;
- (b) As an abelian group R is the direct sum of R_1, \dots, R_n ;
- (c) There exist pairwise orthogonal central idempotents e_1, \dots, e_n with $1 = e_1 + \dots + e_n$, and $R_i = Re_i$, $i = 1, \dots, n$.

Lemma 2.3. (See [12, Theorem 21.28].) Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I . Then there exists an idempotent e in uR such that $e - u \in I$.

Lemma 2.4. (See [8, Lemma 2.4].) Let N be a minimal submodule of M and let $Ann(M)$ be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.

Proposition 2.5. Let $R/Ann(M)$ be an Artinian ring and let M be a finitely generated module. Then every nonzero proper submodule N of M is a vertex in $AG(M)$.

Proof. Let N be a non-zero submodule of M . So there exists a maximal submodule K of M such that $N \subseteq K$. Hence we have $(0 :_M (K : M)) \subseteq (0 :_M (N : M))$. Since $R/Ann(M)$ is an Artinian ring, $(K : M)$ is a minimal prime ideal containing $Ann(M)$. Thus $(K : M) \in Ass(M)$. It follows that $(K : M) = (0 : m)$ for some $0 \neq m \in M$. Therefore $N(Rm) = (0)$, as desired. □

Lemma 2.6. Let $M = M_1 \times M_2$, where $M_1 = eM$, $M_2 = (1 - e)M$, and e ($e \neq 0, 1$) is an idempotent element of R . If $AG(M)$ is a triangle-free graph, then one of the following statements holds.

- (a) Both M_1 and M_2 are prime R -modules.

- (b) One M_i is a prime module for $i = 1, 2$ and the other one is a module with a unique non-trivial submodule.

Moreover, $AG(M)$ has no cycle if and only if either $M = F \times S$ or $M = F \times D$, where F is a simple module, S is a module with a unique non-trivial submodule, and D is a prime module.

Proof. If none of M_1 and M_2 is a prime module, then there exist $r \in R_i$ ($R_1 = Re$ and $R_2 = R(1 - e)$), $0 \neq m_i \in M_i$ with $r_i m_i = 0$, and $r_i \notin \text{Ann}_{R_i}(M_i)$ for $i = 1, 2$. So $r_1 M_1 \times (0)$, $(0) \times r_2 M_2$, and $R_1 m_1 \times R_2 m_2$ form a triangle in $AG(M)$, a contradiction. Thus without loss of generality, one can assume that M_1 is a prime module. We prove that $AG(M_2)$ has at most one vertex. On the contrary suppose that $\{N, K\}$ is an edge of $AG(M_2)$. Therefore, $M_1 \times (0)$, $(0) \times N$, and $(0) \times K$ form a triangle, a contradiction. If $AG(M_2)$ has no vertex, then M_2 is a prime module and so part (a) occurs. If $AG(M_2)$ has exactly one vertex, then by [7, Theorem 3.6] and Proposition 2.5, we obtain part (b). Now, suppose that $AG(M)$ has no cycle. If none of M_1 and M_2 is a simple module, then choose non-trivial submodules N_i in M_i for some $i = 1, 2$. So $N_1 \times (0)$, $(0) \times N_2$, $M_1 \times (0)$, and $(0) \times M_2$ form a cycle, a contradiction. The converse is trivial. \square

Theorem 2.7. *If $AG(M)$ is a tree, then either $AG(M)$ is a star graph or $AG(M) \cong P_4$. Moreover, $AG(M) \cong P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule.*

Proof. If M is a vertex of $AG(M)$, then there exists only one vertex N such that $\text{Ann}(M) = (N : M)$ and since $AG(M)^*$ is an empty subgraph, hence $AG(M)$ is a star graph. Therefore we may assume that M is not a vertex of $AG(M)$. Suppose that $AG(M)$ is not a star graph. Then $AG(M)$ has at least four vertices. Obviously, there are two adjacent vertices N and K of $AG(M)$ such that $|V(N) \setminus \{K\}| \geq 1$ and $|V(K) \setminus \{N\}| \geq 1$. Let $V(N) \setminus \{K\} = \{N_i\}_{i \in \Lambda}$ and $V(K) \setminus \{N\} = \{K_j\}_{j \in \Gamma}$. Since $AG(M)$ is a tree, we have $V(N) \cap V(K) = \emptyset$. By [7, Theorem 3.4], $\text{diam}(AG(M)) \leq 3$. So every edge of $AG(M)$ is of the form $\{N, K\}$, $\{N, N_i\}$ or $\{K, K_j\}$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, consider the following claims:

Claim 1. Either $N^2 = (0)$ or $K^2 = (0)$. Pick $p \in \Lambda$ and $q \in \Gamma$. Since $AG(M)$ is a tree, $N_p K_q$ is a vertex of $AG(M)$. If $N_p K_q = N_u$, for some $u \in \Lambda$, then $KN_u = (0)$, a contradiction. If $N_p K_q = K_v$, for some $v \in \Gamma$, then $NK_v = (0)$, a contradiction. If $N_p K_q = N$ or $N_p K_q = K$, then $N^2 = (0)$ or $K^2 = (0)$, respectively and the claim is proved.

Here, without loss of generality, we suppose that $N^2 = (0)$. Clearly, $(N : M)M \not\subseteq K$ and $(K : M)M \not\subseteq N$.

Claim 2. Our claim is to show that N is a minimal submodule of M and $K^2 \neq (0)$. To see that, first we show that for every $0 \neq m \in N$, $Rm = N$. Assume that $0 \neq m \in N$ and $Rm \neq N$. If $Rm = K$, then $K \subseteq N$, a contradiction. Thus $Rm \neq K$, and the induced subgraph of $AG(M)$ on N , K , and Rm is K_3 , a contradiction. So $Rm = N$. This implies that N is a minimal submodule of M . Now if $K^2 = (0)$, then we obtain the induced subgraph on N , K , and $(N : M)M + (K : M)M$ is K_3 , a contradiction. Thus $K^2 \neq (0)$, as desired.

Claim 3. For every $i \in \Lambda$ and every $j \in \Gamma$, $N_i \cap K_j = N$. Let $i \in \Lambda$ and $j \in \Gamma$. Since $N_i \cap K_j$ is a vertex and $N(N_i \cap K_j) = K(N_i \cap K_j) = (0)$, either $N_i \cap K_j = N$ or $N_i \cap K_j = K$. If $N_i \cap K_j = K$, then $K^2 = (0)$, a contradiction. Hence $N_i \cap K_j = N$ and the claim is proved.

Claim 4. We complete the claim by showing that M has exactly two minimal submodules N and K . Let L be a non-zero submodule properly contained in K . Since $NL \subseteq NK = (0)$, either $L = N$ or $L = N_i$ for some $i \in \Lambda$. So by the Claim 3, $N \subseteq L \subseteq K$, a contradiction. Hence K is a minimal submodule of M . Suppose that L' is another minimal submodule of M . Since N and K both are minimal submodules, we deduce that $NL' = KL' = (0)$, a contradiction. So the claim is proved.

Now by the Claims 2 and 4, $K^2 \neq (0)$ and K is a minimal submodule of M . Then by Lemma 2.4, $K = eM$ for some idempotent $e \in R$. Now we have $M \cong eM \times (1-e)M$. By Lemma 2.6, we deduce that either $M = F \times S$ and $AG(M) \cong P_4$ or $R = F \times D$ and $AG(M)$ is a star graph. Conversely, we assume that $M = F \times S$. Then $AG(M)$ has exactly four vertices $(0) \times S$, $F \times (0)$, $(0) \times N$, and $F \times N$. Thus $AG(M) \cong P_4$ with the vertices $(0) \times S$, $F \times (0)$, $(0) \times N$, and $F \times N$. \square

Theorem 2.8. *Let R be an Artinian ring and $AG(M)$ is a bipartite graph. Then either $AG(M)$ is a star graph or $AG(M) \cong P_4$. Moreover, $AG(M) \cong P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule.*

Proof. First suppose that R is not a local ring. Hence by [9, Theorem 8.9], $R = R_1 \times \dots \times R_n$, where R_i is an Artinian local ring for $i = 1, \dots, n$. By Lemma 2.2 and Proposition 2.1, since $AG(M)$ is a bipartite graph, we have $n = 2$ and hence $M \cong M_1 \times M_2$. If M_1 is a prime module, then it is easy to see that M_1 is a vector space over $R/\text{Ann}(M_1)$ and so is a semisimple R -module. Hence by Lemma 2.6 and Theorem 2.7, we deduce that either $AG(M)$ is isomorphic to P_2 or P_4 . Now we assume that R is an Artinian local ring. Let m be the unique maximal ideal of R and k be a natural number such that $m^k M = 0$ and $m^{k-1} M \neq 0$. Clearly, $m^{k-1} M$ is adjacent to every other vertex of $AG(M)$ and so $AG(M)$ is a star graph. \square

Proposition 2.9. Assume that $\text{Ann}(M)$ is a nil ideal of R .

- (a) If $AG(M)$ is a finite bipartite graph, then either $AG(M)$ is a star graph or $AG(M) \cong P_4$.
- (b) If $AG(M)$ is a regular graph of finite degree, then $AG(M)$ is a complete graph.

Proof. (a). If M is a vertex of $AG(M)$, then $AG(M)$ has only one vertex N such that $\text{Ann}(M) = (N : M)$ and since $AG(M)^*$ is an empty subgraph, $AG(M)$ is a star graph. Thus we may assume that M is not a vertex of $AG(M)$ and hence by [7, Theorem 3.3], M is not a prime module. Therefore [7, Theorem 3.6] follows that $R/\text{Ann}(M)$ is an Artinian ring. If $(R/\text{Ann}(M), m/\text{Ann}(M))$ is a local ring, then there exists a natural number k such that $m^k M = 0$ and $m^{k-1} M \neq 0$. Clearly, $m^{k-1} M$ is adjacent to every other vertex of $AG(M)$ and so $AG(M)$ is a star graph. Otherwise, by [9, Theorem 8.9] and Lemma 2.2, there exist pairwise orthogonal central idempotents modulo $\text{Ann}(M)$. By Lemma 2.3, it is easy to see

that $M \cong eM \times (1 - e)M$, where e is an idempotent element of R and Lemma 2.6 implies that $AG(M)$ is a star graph or $AG(M) \cong P_4$.

(b). If M is a vertex of $AG(M)$, since $AG(M)$ is a regular graph, then $AG(M)$ is a complete graph. Hence we may assume that M is not a vertex of $AG(M)$. So M is not a prime module, and hence $rm = 0$ such that $0 \neq m \in M$, $r \notin \text{Ann}(M)$. It is easy to see that $(rM)(0 :_M r) = (0)$. If the set of R -submodules of rM (resp., $(0 :_M r)$) is infinite, then $(0 :_M r)$ (resp., rM) has infinite degree, a contradiction. Thus rM and $(0 :_M r)$ have finite length. Since $rM \cong M/(0 :_M r)$, M has finite length so that $R/\text{Ann}(M)$ is an Artinian ring. As in the proof of part (a), $M \cong M_1 \times M_2$. If M_1 has one non-trivial submodule N , then $\deg((0) \times M_2) > \deg(N \times M_2)$ and this contradicts the regularity of $AG(M)$. Hence, M_1 is a simple module. Similarly, M_2 is a simple module. So $AG(M) \cong K_2$. Now suppose that $(R/\text{Ann}(M), m/\text{Ann}(M))$ is an Artinian local ring. Now as we have seen in part (a), there exists a natural number k such that $m^{k-1}M$ is adjacent to all other vertices and we deduce that $AG(M)$ is a complete graph. \square

Let S be a multiplicatively closed subset of R . A non-empty subset S^* of M is said to be S -closed if $se \in S^*$ for every $s \in S$ and $e \in S^*$. An S -closed subset S^* is said to be saturated if the following condition is satisfied: whenever $ae \in S^*$ for $a \in R$ and $e \in M$, then $a \in S$ and $e \in S^*$.

We need the following result due to Chin-Pi Lu.

Theorem 2.10. (See [16, Theorem 4.7].) *Let $M = Rm$ be a cyclic module. Let S^* be an S -closed subset of M relative to a multiplicatively closed subset S of R , and N a submodule of M maximal in $M \setminus S^*$. If S^* is saturated, then ideal $(N : M)$ is maximal in $R \setminus S$ so that N is prime in M .*

Theorem 2.11. *If M is a cyclic module, $\text{Ann}(M)$ is a nil ideal, and $|\text{Min}(M)| \geq 3$, then $AG(M)$ contains a cycle.*

Proof. If $AG(M)$ is a tree, then by Theorem 2.7, either $AG(M)$ is a star graph or $M \cong F \times S$, where F is a simple module and S has a unique non-trivial submodule. The latter case is impossible because $|\text{Min}(F \times S)| = 2$. Suppose that $AG(M)$ is a star graph and N is the center of star. Clearly, one can assume that N is a minimal submodule of M . If $N^2 \neq (0)$, then by Lemma 2.4, there exists an idempotent $e \in R$ such that $N = eM$ so that $M \cong eM \times (1 - e)M$. Now by Proposition 2.1 and Lemma 2.6, we conclude that $|\text{Min}(M)| = 2$, a contradiction. Hence $N^2 = 0$. Thus one may assume that $N = Rm$ and $(Rm)^2 = (0)$. Suppose that P_1 and P_2 are two distinct minimal prime submodules of M . Since $(Rm)^2 = (0)$, we have $(Rm : M)^2 \subseteq \text{Ann}(M) \subseteq (P_i : M)$, $i = 1, 2$. So $(Rm : M)M = Rm \subseteq P_i$, $i = 1, 2$. Hence $m \in P_i$, $i = 1, 2$. Choose $z \in (P_1 : M) \setminus (P_2 : M)$ and set $S_1 = \{1, z, z^2, \dots\}$, $S_2 = M \setminus P_1$, and $S^* = S_1 S_2$. If $0 \notin S^*$, then $\Sigma = \{N < M \mid N \cap S^* = \emptyset\}$ is not empty. Then Σ has a maximal element, say N . Hence by Theorem 2.10, N is a prime submodule of M . Since $N \subseteq P_1$, we have $N = P_1$, a contradiction because $z \notin (N : M)$. So $0 \in S^*$. Therefore, there exist positive integer k and $m' \in S_2$ such that $z^k m' = 0$. Now consider the submodules (m) , (m') , and $z^k M$. It is clear that $(m) \neq (m')$ and $(m) \neq z^k M$. If $(m) = z^k M$, then $z \in (P_2 : M)$, a contradiction. Thus (m) , (m') , and $z^k M$ form a triangle in $AG(M)$, a contradiction. Hence $AG(M)$ contains a cycle. \square

Theorem 2.12. *Suppose that M is a cyclic module, $\text{rad}_M(0) \neq (0)$, and $\text{Ann}(M)$ is a nil ideal. If $|\text{Min}(M)| = 2$, then either $\text{AG}(M)$ contains a cycle or $\text{AG}(M) \cong P_4$.*

Proof. A similar argument to the proof of Theorem 2.11 shows that either $\text{AG}(M)$ contains a cycle or $M \cong F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule. The latter case implies that $\text{AG}(M) \cong P_4$ (note that $\text{rad}_{F \times D}(0) = (0)$, where F is a simple module and D is a prime module). \square

We recall that $N < M$ is said to be a semiprime submodule of M if for every ideal I of R and every submodule K of M , $I^2K \subseteq N$ implies that $IK \subseteq N$. Further M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [21]).

Theorem 2.13. *Let S be a multiplicatively closed subset of R containing no zero-divisors on finitely generated module M . Then $\text{cl}(\text{AG}(M_S)) \leq \text{cl}(\text{AG}(M))$. Moreover, $\text{AG}(M_S)$ is a retract of $\text{AG}(M)$ if M is a semiprime module. In particular, $\text{cl}(\text{AG}(M_S)) = \text{cl}(\text{AG}(M))$, whenever M is a semiprime module.*

Proof. Consider a vertex map $\phi : V(\text{AG}(M)) \rightarrow V(\text{AG}(M_S))$, $N \rightarrow N_S$. Clearly, $N_S \neq K_S$ implies $N \neq K$ and $NK = (0)$ if and only if $N_S K_S = (0)$. Thus ϕ is surjective and hence $\text{cl}(\text{AG}(M_S)) \leq \text{cl}(\text{AG}(M))$. In what follows, we assume that M is a semiprime module. If $N \neq K$ and $NK = (0)$, then we show that $N_S \neq K_S$. Without loss of generality we can assume that M is not a vertex of $\text{AG}(M)$ and on the contrary suppose that $N_S = K_S$. Then $N_S^2 = N_S K_S = (NK)_S = (0)$ and so $N^2 = (0)$, a contradiction. This shows that the map ϕ is a graph homomorphism. Now, for any vertex N_S of $\text{AG}(M_S)$, we can choose the fixed vertex N of $\text{AG}(M)$. Then ϕ is a retract (graph) homomorphism which clearly implies that $\text{cl}(\text{AG}(M_S)) = \text{cl}(\text{AG}(M))$ under the assumption. \square

Corollary 2.14. *If M is a finitely generated semiprime module, then $\text{cl}(\text{AG}(T(M))) = \text{cl}(\text{AG}(M))$, where $T = R \setminus Z(M)$.*

Since the chromatic number $\chi(G)$ of a graph G is the least positive integer r such that there exists a retract homomorphism $\psi : G \rightarrow K_r$, the following corollaries follow directly from the proof of Theorem 2.13.

Corollary 2.15. *Let S be a multiplicatively closed subset of R containing no zero-divisors on finitely generated module M . Then $\chi(\text{AG}(M_S)) \leq \chi(\text{AG}(M))$. Moreover, if M is a semiprime module, then $\chi(\text{AG}(M_S)) = \chi(\text{AG}(M))$.*

Corollary 2.16. *If M is a finitely generated semiprime module, then $\chi(\text{AG}(T(M))) = \chi(\text{AG}(M))$, where $T = R \setminus Z(M)$.*

Eben Matlis in [18, Proposition 1.5], proved that if $\{p_1, \dots, p_n\}$ is a finite set of distinct minimal prime ideals of R and $S = R \setminus \cup_{i=1}^n p_i$, then $R_{p_1} \times \dots \times R_{p_n} \cong R_S$. In [20], this result was generalized to finitely generated multiplication modules. In Theorem 2.18, we use this generalization for a cyclic module.

Theorem 2.17. *(See [20, Theorem 3.11].) Let $\{P_1, \dots, P_n\}$ be a finite set of distinct minimal prime submodules of finitely generated multiplication module M and $S = R \setminus \cup_{i=1}^n (P_i : M)$. Then $M_{p_1} \times \dots \times M_{p_n} \cong M_S$, where $p_i = (P_i : M)$ for $1 \leq i \leq n$.*

Theorem 2.18. *Let M be a cyclic module and $\{P_1, \dots, P_n\}$ be a finite set of distinct minimal prime submodules of M . Then there exists a clique of size n .*

Proof. Let M be a cyclic module and $S = R \setminus \cup_{i=1}^n p_i$, where $p_i = (P_i : M)$ for $1 \leq i \leq n$. Then since M is a multiplication module, by Theorem 2.17, there exists an isomorphism $\phi : M_{p_1} \times \dots \times M_{p_n} \longrightarrow M_S$. Let $M = Rm, e_i = (0, \dots, 0, m/1, \dots, 0, \dots, 0)$ and $\phi(e_i) = n_i/t_i$, where $m \in M, 1 \leq i \leq n$, and $m/1$ is in the i -th position of e_i . Consider the principal submodules $N_i = (n_i/t_i) = (n_i/1)$ in the module M_S . By Lemma 2.2 and Proposition 2.1, the product of submodules $(0) \times \dots \times (0) \times (m/1)R_{p_i} \times (0) \times \dots \times (0)$ and $(0) \times \dots \times (0) \times (m/1)R_{p_j} \times (0) \times \dots \times (0)$ are zero, $i \neq j$. Since ϕ is an isomorphism, there exists $t_{ij} \in S$ such that $t_{ij}r_i n_j = 0$, for every $i, j, 1 \leq i < j \leq n$, where $n_i = r_i m$ for some $r_i \in R$. Let $t = \prod_{1 \leq i < j \leq n} t_{ij}$. We show that $\{(tn_1), \dots, (tn_n)\}$ is a clique of size n in $AG(M)$. For every $i, j, 1 \leq i < j \leq n$, $(Rtn_i)(Rtn_j) = (Rtn_j : M)Rtn_i = (Rtn_j : M)tr_i M = tr_i Rtn_j = (0)$. Since $(tn_i)_S = (n_i/1) = N_i$, we deduce that (tn_i) are distinct non-trivial submodules of M . \square

Corollary 2.19. For every cyclic module M , $cl(AG(M)) \geq |Min(M)|$ and if $|Min(M)| \geq 3$, then $gr(AG(M)) = 3$.

Theorem 2.20. *Let M be a cyclic module and $rad_M(0) = (0)$. Then $\chi(AG(M)) = cl(AG(M)) = |Min(M)|$.*

Proof. If $|Min(M)| = \infty$, then by Corollary 2.19, there is nothing to prove. Thus suppose that $|Min(M)| = \{P_1, \dots, P_n\}$, for some positive integer n . Let $p_i = (P_i : M)$ and $S = R \setminus \cup_{i=1}^n p_i$. By Theorem 2.17, we have $M_{p_1} \times \dots \times M_{p_n} \cong M_S$. Clearly, $cl(AG(M_S)) \geq n$. Now we show that $\chi(AG(M_S)) \leq n$. By [15, Corollary 3], $P_i R_{p_i}$ is the only prime submodule of M and since $rad_M(0) = (0)$, every M_{p_i} is a simple R_{p_i} -module. Define the map $C : V(AG(M_S)) \longrightarrow \{1, 2, \dots, n\}$ by $C(N_1 \times \dots \times N_n) = \min\{i \mid N_i \neq (0)\}$. Since each M_{p_i} is a simple module, c is a proper vertex coloring of $AG(M_S)$. Thus $\chi(AG(M_S)) \leq n$ and so $\chi(AG(M_S)) = cl(AG(M_S)) = n$. Since $rad_M(0) = (0)$, it is easy to see that $S \cap Z(M) = \emptyset$. Now by theorem 2.13 and Corollary 2.15, we obtain the desired. \square

Theorem 2.21. *For every module M , $cl(AG(M)) = 2$ if and only if $\chi(AG(M)) = 2$.*

Proof. For the first assertion, we use the same technique in [3, Theorem 13]. Let $cl(AG(M)) = 2$. On the contrary assume that $AG(M)$ is not bipartite. So $AG(M)$ contains an odd cycle. Suppose that $C := N_1 - N_2 - \dots - N_{2k+1} - N_1$ be a shortest odd cycle in $AG(M)$ for some natural number k . Clearly, $k \geq 2$. Since C is a shortest odd cycle in $AG(M)$, $N_3 N_{2k+1}$ is a vertex. Now consider the vertices N_1, N_2 , and $N_3 N_{2k+1}$. If $N_1 = N_3 N_{2k+1}$, then $N_4 N_1 = (0)$. This implies that $N_1 - N_4 - \dots - N_{2k+1} - N_1$ is an odd cycle, a contradiction. Thus $N_1 \neq N_3 N_{2k+1}$. If $N_2 = N_3 N_{2k+1}$, then we have $C_3 = N_2 - N_3 - N_4 - N_2$, again a contradiction. Hence $N_2 \neq N_3 N_{2k+1}$. It is easy to check N_1, N_2 , and $N_3 N_{2k+1}$ form a triangle in $AG(M)$, a contradiction. The converse is clear. \square

The radical of I , defined as the intersection of all prime ideals containing I , denoted by \sqrt{I} . Before stating the next theorem, we recall that if M is a finitely generated module, then $\sqrt{(Q : M)} = (rad(Q) : M)$, where $Q < M$ (see [14] and [17, Proposition 2.3]). Also, we know that if M is a finitely generated module, then

for every prime ideal p of R with $p \supseteq \text{Ann}(M)$, there exists a prime submodule P of M such that $(P : M) = p$ (see [15, Theorem 2]).

Theorem 2.22. *Assume that M is a finitely generated module, $\text{Ann}(M)$ is a nil ideal, and $|\text{Min}(M)| = 1$. If $AG(M)$ is a triangle-free graph, then $AG(M)$ is a star graph.*

Proof. Suppose first that P is the unique minimal prime submodule of M . Since M is not a vertex of $AG(M)$, hence $Z(M) \neq (0)$. So there exist non-zero elements $r \in R$ and $m \in M$ such that $rm = 0$. It is easy to see that rM and Rm are vertices of $AG(M)$ because $(rM)(Rm) = 0$. Since $AG(M)$ is triangle-free, Rm or rM is a minimal submodule of M . Without loss of generality, we can assume that Rm is a minimal submodule of M so that $(Rm)^2 = (0)$ (if rM is a minimal submodule of M , then there exists $0 \neq m' \in M$ such that $rM = Rm'$). We claim that Rm is the unique minimal submodule of M . On the contrary, suppose that K is another minimal submodule of M . So either $K^2 = K$ or $K^2 = 0$. If $K^2 = K$, then by Lemma 2.4, $K = eM$ for some idempotent element $e \in R$ and hence $M \cong eM \times (1 - e)M$. This implies that $|\text{Min}(M)| > 1$, a contradiction. If $K^2 = 0$, then we have $C_3 = K - (K : M)M + (Rm : M)M - Rm - K$, a contradiction. So Rm is the unique minimal submodule of M . Let $V_1 = N((Rm))$, $V_2 = V(AG(M)) \setminus V_1$, $A = \{K \in V_1 | (Rm) \subseteq K\}$, $B = V_1 \setminus A$, and $C = V_2 \setminus \{Rm\}$. We prove that $AG(M)$ is a bipartite graph with parts V_1 and V_2 . We may assume that V_1 is an independent set because $AG(M)$ is triangle-free. We claim that one end of every edge of $AG(M)$ is adjacent to Rm and another end contains Rm . To prove this, suppose that $\{N, K\}$ is an edge of $AG(M)$ and $Rm \neq N$, $Rm \neq K$. Since $N(Rm) \subseteq Rm$, by the minimality of Rm , either $N(Rm) = (0)$ or $Rm \subseteq N$. The latter case follows that $K(Rm) = (0)$. If $N(Rm) = (0)$, then $K(Rm) \neq (0)$ and hence $Rm \subseteq K$. So our claim is proved. This gives that V_2 is an independent set and $N(C) \subseteq V_1$. Since every vertex of A contains Rm and $AG(M)$ is triangle-free, all vertices in A are just adjacent to Rm and so by [7, Theorem 3.4], $N(C) \subseteq B$. Since one end of every edge is adjacent to Rm and another end contains Rm , we also deduce that every vertex of C contains Rm and so every vertex of $A \cup V_2$ contains Rm . Note that if $Rm = P$, then one end of each edge of $AG(M)$ is contained in Rm and since Rm is a minimal submodule of M , $AG(M)$ is a star graph with center $Rm = P$. Now, suppose that $P \neq Rm$. We claim that $P \in A$. Since $Rm \subseteq P$, it suffices to show that $(Rm)P = (0)$. To see this, let $r \in (P : M)$. We prove that $rm = 0$. Clearly, $(Rrm) \subseteq Rm$. If $rm = 0$, then we are done. Thus $Rrm = Rm$ and so $m = rsm$ for some $s \in R$. We have $m(1 - rs) = 0$. By [15, Theorem 2], we have $\text{Nil}(R) = (P : M)$ (note that $\sqrt{\text{Ann}(M)} = (\text{rad}(0) : M) = (P : M)$). Therefore $1 - rs$ is unit, a contradiction, as required. Since $N(C) \subseteq B$, if $B = \emptyset$, then $C = \emptyset$ and so $AG(M)$ is a star graph with center Rm . It remains to show that $B \neq \emptyset$. Suppose that $K \in B$ and consider the vertex $K \cap P$ of $AG(M)$. Since every vertex of $A \cup V_2$ contains Rm , yields $K \cap P \in B$. Pick $0 \neq m' \in K \cap P$. Since $AG(M)$ is triangle-free, one can find an element $m'' \in Rm'$ such that Rm'' is a minimal submodule of M and $(Rm'')^2 = (0)$. Since Rm is the unique minimal submodule of M , we have $Rm = Rm'' \subseteq Rm'$. Thus $Rm \subseteq K \cap P$, a contradiction. So $B = \emptyset$ and we are done. Hence $AG(M)$ is a star graph whose center is Rm , as desired. \square

Corollary 2.23. Assume that M is a finitely generated module, $\text{Ann}(M)$ is a nil ideal, and $|\text{Min}(M)| = 1$. If $\text{AG}(M)$ is a bipartite graph, then $\text{AG}(M)$ is a star graph.

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